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A Modified Tau Spectral Method That Eliminates Spurious Eigenvalues

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A modified tau spectral method is presented which eliminates the spurious eigenvalues produced by the usual tau method. The modified tau method essentially involves an appropriate factorization of the differential operators in the eigenvalue problem. It is developed for eigenvalue problems posed as single differential equations or as systems of such equations. Several eigenvalue problems are solved using both the usual and modified Chebyshev–tau methods, including the Orr–Sommerfeld stability equation for plane Poiseuille flow. The convergence of the modified tau method is shown to be at least as rapid as that of the usual tau method. The use of the tau coefficients in indicating convergence is also discussed.

1. INTRODUCTION

Spectral methods offer useful alternatives to finite difference and finite element methods for solving differential equations and eigenvalue problems. Advantages of spectral methods include the production of a global solution, rapid convergence, and, in some methods, avoidance of the Gibbs phenomenon at domain boundaries [1–3]. A particularly useful spectral method is the Chebyshev–tau method, in which Chebyshev polynomials are used in the tau method of Lanczos [4]. Application of the Chebyshev-tau method to eigenvalue problems, especially to hydrodynamic stability problems, has been hindered by the observation that the method can generate spurious eigenvalues; in a problem where hydrodynamic stability is indicated by eigenvalues with negative real parts, spurious eigenvalues

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with large positive real parts may appear even in a regime where the solution is known to be stable [3, 5, 6]. These spurious eigenvalues appear even when a pseudo-Chebyshev–Galerkin approach (in which the basis functions satisfy the boundary conditions and the test functions are Chebyshev polynomials) [6] is used because they arise from the modeling of the differential operator as well as from the boundary conditions. Thus using basis functions that satisfy the boundary conditions will not remove all the spurious eigenvalues, though it may improve convergence somewhat [6]. A Chebyshev-Galerkin approach recently has been proposed which does eliminate spurious eigenvalues [7], although the method is less general than the modified tau method proposed here.

In the following sections the usual tau method will be described and a modified tau method will be presented which eliminates spurious eigenvalues. The modified method converges at least as rapidly as the usual method. In addition, the use of the often neglected tau coefficients as convergence indicators will be discussed and a variety of examples using the usual and modified Chebyshev–tau method will be presented, including the Orr–Sommerfeld stability problem for plane Poiseuille flow. Extensions of the modified tau method to systems of equations will also be discussed. An appendix is included which contains several useful relationships involving Chebyshev polynomials. Although available, in part, from other sources (e.g., [3]) the relationships are included here so that a user of the method can have all necessary information in one location.

2. The Tau Method

The tau method was first proposed by Lanczos [4] and its use with Chebyshev polynomials (Chebyshev functions of the first kind) was later developed extensively by Fox [8] and was applied and advocated by Orszag for a wide variety of problems [1–3]. The tau method uses a truncated series expansion in a set of complete functions as an approximation for the solution of an ordinary differential equation; the method can also be applied to partial differential equations, integral equations, and integro-differential equations.

To illustrate the use of the tau method, consider an approximate, truncated series solution to the ordinary differential equation

$$L\{u\}=0, \qquad -1<x<1, \tag{2.1}$$

$$B_i\{u(-1)\}=B_j\{u(1)\}=0, \qquad i+j=N_b, \tag{2.2}$$

where L is a linear ordinary differential operator of order $n=N_b$, u is an unknown function in the independent variable x, and B_i and B_j represent linear operators such that $B_i\{u(-1)\}=0$ and $B_j\{u(1)\}=0$ represent the boundary conditions. The subscripts i and j index the number of boundary conditions applied at $x=-1$ and $x=1$, respectively. (It is assumed here that differential equation (2.1) contains no singularities in the interval of interest; if it does, then the problem must be treated in the complex domain as described in [9], but the modified method presented in Section 3 can still be applied.) Let u be approximated by $u(x, N)$,

$$u(x, N) = \sum_{k=0}^{N+N_b} a_k f_k(x) \tag{2.3}$$

where the a_k's are unknown coefficients and the f_k's are the functions from the complete set which has inner product symbolized by $\langle \cdot, \cdot \rangle$, and which are orthogonal with respect to this inner product. Substituting (2.3) into (2.1) and using the inner product for the functions f_k, $k = 0, 1, ..., N$, yields a system of $N + 1$ algebraic equations for the unknown coefficients a_k, $k = 0, 1, ..., N + N_b$. The remaining N_b equations are found by substitution of (2.3) into (2.2) directly so that the following system of $N + N_b + 1$ equations, linear in the a_k's, is produced:

$$\langle L\{u(x, N)\}, f_k \rangle = 0, \qquad k = 0, 1, ..., N, \tag{2.4a}$$

$$\sum_{k=0}^{N+N_b} B_i\{f_k(-1)\}\, a_k = \sum_{k=0}^{N+N_b} B_j\{f_k(1)\}\, a_k = 0, \qquad i + j = N_b. \tag{2.4b}$$

If the problem is an eigenvalue problem, then the unknown eigenvalue of course appears in the corresponding matrix equation. Equation (2.4a, b) can be solved using standard matrix solver packages like those in the EISPACK or IMSL libraries.

The tau method solves the following modification of problem (2.1, 2) *exactly*:

$$L\{u\} = \sum_{k=1}^{N_b} \tau_k f_{N+k}(x), \tag{2.5}$$

$$\sum_{k=0}^{N+N_b} B_i\{f_k(-1)\}\, a_k = \sum_{k=0}^{N+N_b} B_j\{f_k(1)\}\, a_k = 0, \qquad i + j = N_b, \tag{2.6a, b}$$

where the coefficients τ_k for $k = 1, ..., N_b$ are unknowns whose values depend ultimately on the boundary conditions. Using the inner product for the functions f_k differential equation (2.5) is reduced to the system of equations

$$\langle L\{u(x, N)\}, f_k \rangle = 0, \qquad k = 0, 1, ..., N, \tag{2.7a}$$

$$\langle L\{u(x, N)\}, f_{N+k} \rangle = \tau_k \langle f_{N+k}, f_{N+k} \rangle, \qquad k = 1, ..., N_b. \tag{2.7b}$$

Equation (2.7a) and boundary conditions (2.6a, b) are then solved to yield the values of the coefficients a_k. If desired, the τ coefficients τ_k, $k = 1, ..., N_b$, can then be computed from Eq. (2.7b). Fox [8] has shown that the tau coefficients can be used to place error bounds on the approximate solution $u(x, N)$. When the Chebyshev–tau method is used, the error introduced by the tau method will be small if the magnitudes of the tau coefficients are small, since the Chebyshev polynomials are bounded in absolute value by one on the interval $[-1, 1]$. In common practice the tau coefficients are not computed.

In the Chebyshev–tau method, the Chebyshev polynomials $T_n(x)$ (see the Appendix) are used as the complete set of expansion functions, with the inner product in which Chebyshev polynomials of different orders are orthogonal. The formulae given in the Appendix are used to expand products of Chebyshev polynomials and derivatives of Chebyshev polynomials as expansions in Chebyshev polynomials. For example, if a function $g(x)$ and its first derivative $g'(x)$ have series expansions in terms of Chebyshev polynomials

$$g(x) = \sum_{n=0}^{N} b_n T_n(x), \tag{2.8}$$

$$g'(x) = \sum_{n=0}^{N} b_n^{(1)} T_n(x) \tag{2.9}$$

then the coefficients $b_n^{(1)}$ are related to the coefficients b_n by

$$c_n b_n^{(1)} = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{p=N} p b_p, \tag{2.10}$$

$$c_n = \begin{cases} 2, & \text{if} \quad n = 0; \\ 1, & \text{if} \quad n = 1, 2, ... \end{cases} \tag{2.11}$$

The Chebyshev expansion coefficients b_n for a known function $f(x)$ may of course be determined on the interval $[-1, 1]$ by using the orthogonality of the Chebyshev polynomials (see the Appendix):

$$b_n = \frac{2}{\pi c_n} \langle f, T_n \rangle = \frac{2}{\pi c_n} \int_{-1}^{1} f(x)\, T_n(x)(1 - x^2)^{-1/2}\, dx.$$

A convenient way to determine approximate values for the coefficients is to use the fact that the relation

$$f(\cos(k\pi/N)) = \sum_{n=0}^{N} b_n \cos(\pi nk/N), \qquad k = 0, ..., N,$$

is invertible using discrete Fourier transforms [2] as

$$d_n b_n = (2/N) \sum_{k=0}^{N} d_k^{-1} f(\cos(\pi k/N)) \cos(\pi kn/N), \qquad n = 0, ..., N,$$

$$d_n = \begin{cases} 2, & \text{for} \quad n = 0, N; \\ 1, & \text{for} \quad n = 1, ..., N-1. \end{cases}$$

To further illustrate the application of the usual tau method, consider the eigenvalue problem

$$u'''' + Ru''' - su'' = 0, \qquad -1 < x < 1, \tag{2.12}$$

$$u(-1) = u(1) = u'(-1) = u'(1) = 0 \tag{2.13}$$

where u is the unknown function, R is a real parameter, s is an eigenvalue, and a prime denotes differentiation with respect to x. To apply the usual tau method to this problem, a complete set of functions $\{f_k\}$ must first be selected; here Chebyshev polynomials are chosen and so the Chebyshev–tau method is being used. The Chebyshev polynomials $T_n(x)$ are orthogonal with respect to the inner product defined by

$$\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)\, T_m(x)(1-x^2)^{-1/2}\, dx.$$

Each term in the differential equation along with each term in boundary conditions (2.13) is expanded as a truncated series of Chebyshev polynomials as

$$u(x) \approx \sum_{k=0}^{N+4} a_k T_k(x), \tag{2.14}$$

$$a_k^{(4)} + Ra_k^{(3)} - sa_k^{(2)} = 0, \qquad k = 0, ..., N, \tag{2.15}$$

$$\sum_{j=0}^{N+4} (-1)^j a_j = \sum_{j=0}^{N+4} a_j = \sum_{j=0}^{N+4} (-1)^{j+1} j^2 a_j = \sum_{j=0}^{N+4} j^2 a_j = 0, \tag{2.16}$$

$$a_{N+k}^{(4)} + Ra_{N+k}^{(3)} - sa_{N+k}^{(2)} = \tau_k, \qquad k = 1, 2, 3, 4. \tag{2.17}$$

The coefficients $a_k^{(2)}$, $a_k^{(3)}$, and $a_k^{(4)}$ are given by

$$c_k a_k^{(2)} = \sum_{\substack{p=k+2 \\ p+k \text{ even}}}^{N+4} p(p^2-k^2)\, a_p, \tag{2.18}$$

$$c_k a_k^{(3)} = \frac{1}{4} \sum_{\substack{p=k+3 \\ p+k \text{ odd}}}^{N+4} p[p^2(p^2-2) - 2p^2k^2 + (k^2-1)^2]\, a_p, \tag{2.19}$$

$$c_k a_k^{(4)} = \frac{1}{24} \sum_{\substack{p=k+4 \\ p+k \text{ even}}}^{N+4} p[p^2(p^2-4)^2 - 3p^4k^2 + 3p^2k^4 - k^2(k^2-4)^2]\, a_p. \tag{2.20}$$

The system of equations (2.15)–(2.16) can be written in matrix form as

$$\mathbf{Aa} = s\mathbf{Ba} \tag{2.21}$$

where the matrices **A** and **B** have elements given by

$$A_{ni}=\chi_4(n,i)\,F_4(n,i)+R\chi_3(n,i)\,F_3(n,i),\qquad n=0,...,N;\quad i=0,...,N+4, \tag{2.22a}$$

$$F_3(n,i)=\frac{1}{4c_n}\,i[i^2(i^2-2)-2i^2n^2+(n^2-1)^2], \tag{2.22b}$$

$$F_4(n,i)=\frac{1}{24c_n}\,i[i^2(i^2-4)^2-3i^4n^2+3i^2n^4-n^2(n^2-4)^2], \tag{2.22c}$$

$$A_{N+1,i}=(-1)^i,\qquad A_{N+2,i}=1,\qquad A_{N+3,i}=-(-1)^i i^2,\qquad A_{N+4,i}=i^2, \tag{2.22d, e, f, g}$$

$$B_{ni}=i(i^2-n^2)\,\chi_2(n,i),\qquad n=0,...,N;\qquad i=0,...,N+4, \tag{2.22h}$$

$$B_{ni}=0,\qquad n=N+1,...,N+4;\quad i=0,...,N+4, \tag{2.22i}$$

$$\chi_M(n,i)=\begin{cases}0, & \text{if}\quad i<n+M\quad\text{or}\quad n+i+M\quad\text{odd};\\ 1, & \text{if}\quad i\geqslant n+M\quad\text{and}\quad n+i+M\quad\text{even},\end{cases} \tag{2.23}$$

and the column vector **a** is the vector of the unknown coefficients a_n.

If one seeks to solve the matrix eigenvalue problem (2.21) as posed above with an eigenvalue solver package like the IMSL packages EIGRF or EIGZC, spurious eigenvalues are generated by the boundary condition rows owing to the resulting singular nature of the matrix **B**. These eigenvalues are, strictly speaking, infinite in magnitude, and are not true eigenvalues of the problem. This difficulty can be averted by removing the boundary condition rows from the problem which also reduces the order of the resulting matrix problem by four (more generally, by N_b). The elimination of the boundary condition rows can result in significant savings when the eigenvalue problem is posed as a system of differential equations.

Boundary conditions (2.16) imply that only $N+1$ of the $N+N_b+1=N+5$ coefficients a_k are independent, and thus any four of them can be expressed in terms of the remaining coefficients. To do this, the matrices **A** and **B** are partitioned as

$$A_{1ni}=A_{ni},$$

$$B_{1ni}=B_{ni},\qquad n=0,...,N;\quad i=0,...,N, \tag{2.24a, b}$$

$$A_{2ni}=A_{n,i+N+1},$$

$$B_{2ni}=B_{n,i+N+1},\qquad n=0,...,N;\quad i=0,...,3, \tag{2.24c, d}$$

$$A_{3ni}=A_{n+N+1,i},$$

$$B_{3ni}=B_{n+N+1}=0,\qquad n=0,...,3;\quad i=0,...,N, \tag{2.24e, f}$$

$$A_{4ni}=A_{n+N+1,i+N+1},$$

$$B_{4ni}=B_{n+N+1,i+N+1}=0,\qquad n=0,...,3;\quad i=0,...,3. \tag{(2.24g, h}$$

Thus there are two matrix equations

$$\mathbf{A}_1 \mathbf{a}_1 + \mathbf{A}_2 \mathbf{a}_2 = s\mathbf{B}_1 \mathbf{a}_1 + s\mathbf{B}_2 \mathbf{a}_2, \tag{2.25a}$$

$$\mathbf{A}_3 \mathbf{a}_1 + \mathbf{A}_4 \mathbf{a}_2 = \mathbf{0} \tag{2.25b}$$

where $\mathbf{a}_1 = (a_0, a_1, ..., a_N)^{\mathrm{T}}$ and $\mathbf{a}_2 = (a_{N+1}, ..., a_{N+4})^{\mathrm{T}}$. Equation (2.25b) can be solved for $\mathbf{a}_2$ in terms of $\mathbf{a}_1$ as $\mathbf{a}_2 = -\mathbf{A}_4^{-1} \mathbf{A}_3 \mathbf{a}_1$ and the result substituted into Eq. (2.25a) to produce the $[N+1] \times [N+1]$ matrix eigenvalue problem

$$[\mathbf{A}_1 - \mathbf{A}_2 \mathbf{A}_4^{-1} \mathbf{A}_3]\, \mathbf{a}_1 = s[\mathbf{B}_1 - \mathbf{B}_2 \mathbf{A}_4^{-1} \mathbf{A}_3]\, \mathbf{a}_1. \tag{2.26}$$

From (2.22d–g) and (2.24g) it is clear that $\mathbf{A}_4$ is non-singular. Once (2.26) has been solved for the eigenvalue s and the eigenvector $\mathbf{a}_1$ the remaining coefficients $\mathbf{a}_2$ can be computed directly by matrix multiplication as defined in (2.25b).

The tau coefficients can be computed once the eigenvalue s and the original eigenvector $\mathbf{a}$ have been determined from the equations

$$\tau = [\mathbf{C} - s\mathbf{D}]\, \mathbf{a}, \tag{2.27a}$$

$$C_{ni} = \chi_4(n+N, i)\, F_4(n+N, i) + R\chi_3(n+N, i)\, F_3(n+N, i), \tag{2.27b}$$

$$D_{ni} = \chi_2(n+N, i)\, i(i^2 - (n+N)^2), \qquad n = 1, ..., 4; \quad i = 0, ..., N+4, \tag{2.27c}$$

$$\tau = (\tau_1, \tau_2, \tau_3, \tau_4)^{\mathrm{T}} \tag{2.27d}$$

where the functions F_3 and F_4 are defined in Eq. (2.22b, c).

3. The Modified Tau Method

The straightforward application of the tau method presented in the previous section works very well for solving differential equations. As shown in Section 4 and in other works (e.g., [3]), spurious eigenvalues are generated by this method when it is applied to many eigenvalue problems of order greater than two and in general reflects the numerical instability of the method. In some cases the spurious eigenvalues may in fact be solutions to the eigencondition but are not true eigenvalues for the problem. These spurious eigenvalues can be difficult to detect, and if the usual tau method is used some unambiguous criterion is needed to identify them. One detection method involves computing a set of eigenvalues and τ's for a given approximation of order N and to then recompute the same quantities for an approximation of order $N+1$. Those eigenvalues and corresponding τ's which fluctuate wildly as N is varied are spurious. Those eigenvalues with wellbehaved τ coefficients are true eigenvalues. This, of course, is a viable method for simple problems or in problems where one already knows the true eigenvalues (e.g., the Orr–Sommerfeld problem tested in Section 4 and in [2].) In general this method is inefficient, being computationally intensive and expensive (for large problems), and may in fact be inconclusive for relatively small values of N. It is, therefore, preferable not to generate spurious eigenvalues at all. In this section a method is proposed which eliminates spurious eigenvalues. This modified method essentially involves an appropriate factorization of the differential operator which removes the numerical instability. The general idea resembles the stream function–vorticity formulation alluded to in [3], which in itself is insufficient to ensure that spurious eigenvalues are not produced, as shown in [5].

3.1. *An Illustration of the Modified Tau Method*

The modified Chebyshev–tau method will be illustrated for the eigenvalue problem of the previous section, viz.,

$$u'''' + Ru''' - su'' = 0, \qquad -1 < x < 1, \tag{3.1}$$

$$u(-1) = u(1) = u'(-1) = u'(1) = 0 \tag{3.2}$$

where u is the unknown function, R is a real parameter, s is the eigenvalue, and a prime denotes differentiation with respect to x. Application of the method to other problems then will become clear. Problem (3.1)–(3.2) is sufficiently simple to be solved analytically while retaining the essential features necessary to illustrate the application of the modified Chebyshev–tau method.

To apply the modified tau method to (3.1)–(3.2), the differential equation is first written as a system of two second-order ordinary differential equations

$$v'' + Rv' - sv = 0, \qquad v = u''. \tag{3.3a, b}$$

The terms in these differential equations, with those in boundary conditions (3.2), are next expanded with truncated series of Chebyshev polynomials as

$$u(x) \approx u(x, N) = \sum_{n=0}^{N+2} a_n T_n(x), \tag{3.4a}$$

$$v(x) \approx v(x, N) = \sum_{n=0}^{N+2} b_n T_n(x). \tag{3.4b}$$

Each equation is then replaced by the corresponding Chebyshev–tau approximation:

$$b_n^{(2)} + Rb_n^{(1)} - sb_n = 0, \qquad n = 0, ..., N, \tag{3.5a}$$

$$b_{k+N}^{(2)} + Rb_{k+N}^{(1)} - sb_{k+N} = \tau_k, \qquad k = 1, 2, \tag{3.5b}$$

$$a_n^{(2)} = b_n, \qquad n = 0, ..., N, \tag{3.5c}$$

$$a_{k+N}^{(2)} - b_{k+N} = \tilde{\tau}_k, \qquad k = 1, 2, \tag{3.5d}$$

$$\sum_{n=0}^{N+2} (-1)^n a_n = \sum_{n=0}^{N+2} a_n = \sum_{n=0}^{N+2} (-1)^{n+1} n^2 a_n = \sum_{n=0}^{N+2} n^2 a_n = 0 \tag{3.5e–h}$$

where (3.5e–h) represent boundary conditions (3.2), the coefficients $b_n^{(2)}$, $b_n^{(1)}$, and $a_n^{(2)}$ are given by

$$c_n b_n^{(1)} = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{N+2} p b_p, \tag{3.6a}$$

$$c_n b_n^{(2)} = \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{N+2} p(p^2-n^2)\, b_p, \tag{3.6b}$$

$$c_n a_n^{(2)} = \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{N+2} p(p^2-n^2)\, a_p, \tag{3.6c}$$

and c_n is defined in (2.11). Note that since (3.3b) is a definition, $\tilde{\tau}_k = 0$ for $k = 1, 2$ in (3.5d) and so there will be only two tau coefficients, those given by (3.5b). This is verified by computation when the $\tilde{\tau}_k$'s are left as unknown coefficients, resulting in $\tilde{\tau}_k = 0$, $k = 1, 2$.

Since the boundary conditions apply to the function u and not v (and hence involve only the coefficients a_n, $n = 0, ..., N+2$) the b-coefficients will be eliminated from (3.5a, b) by using (3.5c). The procedure for doing this is readily seen by writing (3.5a–c) in matrix notation as

$$\mathbf{B}_1\mathbf{b} + \mathbf{B}_4\mathbf{y} - s\mathbf{b}_1 = \mathbf{0}, \tag{3.7a}$$

$$\mathbf{B}_2\mathbf{b} + \mathbf{B}_5\mathbf{y} - s\mathbf{b}_2 = \mathbf{0}, \tag{3.7b}$$

$$\tau = \mathbf{B}_3\mathbf{b} + \mathbf{B}_6\mathbf{y} - s\mathbf{y}, \tag{3.7c}$$

$$\mathbf{b} = \mathbf{Q}\mathbf{a} \tag{3.7d}$$

where the matrix elements are defined by

$$B_{1ni} = \chi_2(n, i)\, F_2(n, i) + R\chi_1(n, i)\, F_1(n, i), \qquad n = 0, ..., N-2, \quad i = 0, ..., N, \tag{3.8a}$$

$$B_{2ni} = \chi_2(N-1+n, i)\, F_2(N-1+n, i) + R\chi_1(N-1+n, i)\, F_1(N-1+n, i), \qquad n = 0, 1; \quad i = 0, ..., N, \tag{3.8b}$$

$$B_{3ni} = \chi_2(N+1+n, i)\, F_2(N+1+n, i) + R\chi_1(N+1+n, i)\, F_1(N+1+n, i), \qquad n = 0, 1; \quad i = 0, ..., N, \tag{3.8c}$$

$$B_{4ni} = \chi_2(n, N+1+i)\, F_2(n, N+1+i) + R\chi_1(n, N+1+i)\, F_1(n, N+1+i), \qquad n = 0, ..., N-2; \quad i = 0, 1, \tag{3.8d}$$

$$B_{5ni} = \chi_2(N-1+n, N+1+i)\, F_2(N-1+n, N+1+i)$$
$$+ R\chi_1(N-1+n, N+1+i)\, F_1(N-1+n, N+1+i),$$
$$n = 0, 1; \quad i = 0, 1, \tag{3.8e}$$

$$B_{6ni} = \chi_2(N+1+n, N+1+i)\, F_2(N+1+n, N+1+i)$$
$$+ R\chi_1(N+1+n, N+1+i)\, F_1(N+1+n, N+1+i),$$
$$n = 0, 1; \quad i = 0, 1, \tag{3.8f}$$

$$Q_{ni} = \chi_2(n, i)\, F_2(n, i) + R\chi_1(n, i)\, F_1(n, i), \qquad n = 0, ..., N; \quad i = 0, ..., N+2, \tag{3.8g}$$

$$Q_{1ni} = Q_{ni}, \qquad n = 0, ..., N-2; \quad i = 0, ..., N+2, \tag{3.8h}$$

$$Q_{2ni} = Q_{n+N-1,i}, \qquad n = 0, 1; \quad i = 0, ..., N+2, \tag{3.8i}$$

$$F_1(n, i) = 2i/c_n, \qquad F_2(n, i) = i(i^2 - n^2)/c_n \tag{3.8j, k}$$

and where

$$\mathbf{b} = (b_0, ..., b_N)^{\mathrm{T}}, \qquad \mathbf{b}_1 = (b_0, ..., b_{N-2})^{\mathrm{T}} = \mathbf{Q}_1\mathbf{a},$$
$$\mathbf{b}_2 = (b_{N-1}, b_N)^{\mathrm{T}} = \mathbf{Q}_2\mathbf{a}$$
$$\mathbf{y} = (b_{N+1}, b_{N+2})^{\mathrm{T}}, \qquad \tau = (\tau_1, \tau_2)^{\mathrm{T}}, \qquad \mathbf{a} = (a_0, ..., a_{N+2})^{\mathrm{T}}.$$

This partitioning of the matrix equations is shown in Fig. 1.

The objective now is to remove the vector $\mathbf{y}$ from matrix equations (3.7a, c); this is done by solving (3.7b) with (3.7d) for $\mathbf{y}$:

$$\mathbf{y} = -\mathbf{B}_5^{-1}[\mathbf{B}_2\mathbf{Q} - s\mathbf{Q}_2]\,\mathbf{a}. \tag{3.9a}$$

$$\left[\begin{array}{c|c} \mathbf{B}_1 & \mathbf{B}_4 \\ \hline \mathbf{B}_2 & \mathbf{B}_5 \\ \hline \mathbf{B}_3 & \mathbf{B}_6 \end{array}\right] \left[\begin{array}{c} \mathbf{b} \\ \\ \hline \mathbf{y} \end{array}\right] - s \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{y} \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \\ \hline \tau \end{array}\right]$$

$$\left[\begin{array}{c} \mathbf{Q}_1 \\ \hline \mathbf{Q}_2 \end{array}\right] \left[\begin{array}{c} \mathbf{a} \end{array}\right] = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \end{array}\right] = \left[\begin{array}{c} \mathbf{b} \end{array}\right]$$

FIG. 1. Partitioning of the matrix equations (3.7a–d) in Section 3.1.

Substituting (3.9a) in (3.7a) results in

$$[\mathbf{B}_1\mathbf{Q} - \mathbf{B}_4\mathbf{B}_5^{-1}\mathbf{B}_2\mathbf{Q}]\,\mathbf{a} = s[\mathbf{Q} - \mathbf{B}_4\mathbf{B}_5^{-1}\mathbf{Q}_2]\,\mathbf{a} \tag{3.9b}$$

and the unknown τ coefficients are found from

$$\tau = \mathbf{B}_3\mathbf{Q}\mathbf{a} + \mathbf{B}_6\mathbf{y} - s\mathbf{y} \tag{3.9c}$$

after the eigenvalue s and eigenvector $\mathbf{a}$ are computed from (3.9b). Note that (3.7b) is relatively easy to solve since $\mathbf{B}_5$ is a 2×2 matrix and that $\mathbf{B}_5$ is non-singular unless $R = \pm 2[N(N+1)]^{1/2}$. Comparison of (3.9b) with (2.21) reveals the difference in the modeling of the differential operator.

As with the usual tau method, it is useful to remove the boundary condition rows (3.5e–h) from matrix equation (3.9b); the procedure is identical to that used in Section 2.

Numerical examples of the use of the usual and modified Chebyshev-tau methods will be presented in Section 4.

3.2. *The Modified Tau Method for a General Fourth-Order Eigenvalue Problem*

The modified tau method for a class of general fourth-order eigenvalue problems can be represented as follows. Suppose the differential equation is written as a system of two second-order differential equations

$$L_1\{v\} + L_2\{u\} - s[v + L_3\{u\}] = 0, \qquad v = L_4\{u\}, \tag{3.10a, b}$$

$$B_i\{u(-1)\} = B_j\{u(1)\} = 0, \qquad i + j = N_b = 4, \tag{3.10c}$$

where L_k for $k = 1, ..., 4$ are linear second-order differential operators and B_i and B_j, for $i + j = N_b$ are linear operators representing the boundary conditions as in Section 2. Expanding u and v as truncated series of Chebyshev polynomials

$$u(x) \approx \sum_{n=0}^{N+2} a_n T_n(x), \tag{3.11a}$$

$$v(x) \approx \sum_{n=0}^{N+2} b_n T_n(x), \tag{3.11b}$$

(3.10a–c) can be replaced by the system of equations

$$\sum_{i=0}^{N+2} \{B_{ni}b_i + A_{ni}a_i - s[b_i + C_{ni}a_i]\} = 0, \qquad n = 0, ..., N, \tag{3.12a}$$

$$\tau_n = \sum_{i=0}^{N+2} \{B_{n+N,i}\,b_i + A_{n+N,i}a_i - s[b_i + C_{n+N,i}a_i]\}, \qquad n = 1, 2, \tag{3.12b}$$

$$\sum_{i=0}^{N+2} Q_{ni} a_i = b_n, \qquad n = 0, ..., N, \tag{3.12c}$$

$$\sum_{n=0}^{N+2} B_i\{T_n(-1)\}\, a_n = \sum_{n=0}^{N+2} B_j\{T_n(1)\}\, a_n = 0, \qquad i+j=4, \tag{3.12d}$$

where the matrix **B** represents the action of the operator L_1 on v, the matrix **A** represents the action of the operator L_2 on u, the matrix **C** represents the action of the operator L_3 on u, and the matrix **Q** represents the action of the operator L_4 on u. As before, the two tau coefficients associated with (3.12c) are identically zero since this equation is exact, and hence the tau coefficient rows are not shown. Equation (3.12a–c) can be partitioned exactly as was done for the example in Section 3.1, with matrix **B** here partitioned exactly as matrix **B** was there, and matrices **A**, **C**, and **Q** here partitioned exactly as was matrix **Q** in Section 3.1. The b-coefficients are removed as before to yield a matrix problem of the form

$$\mathbf{y} = -\mathbf{B}_5^{-1}[\mathbf{B}_2\mathbf{Q} + \mathbf{A}_2 - s(\mathbf{Q}_2 + \mathbf{C}_2)]\,\mathbf{a}, \tag{3.13a}$$

$$[\mathbf{B}_1 - \mathbf{B}_4\mathbf{B}_5^{-1}\,\mathbf{B}_2]\,\mathbf{Qa} + [\mathbf{A}_1 - \mathbf{B}_4\mathbf{B}_5^{-1}\,\mathbf{A}_2]\,\mathbf{a}$$
$$= s[\mathbf{C}_1 + \mathbf{Q}_1 - \mathbf{B}_4\mathbf{B}_5^{-1}(\mathbf{C}_2 + \mathbf{Q}_2)]\,\mathbf{a}, \tag{3.13b}$$

$$\tau = [\mathbf{B}_3\mathbf{Q} + \mathbf{A}_3]\,\mathbf{a} + \mathbf{B}_6\mathbf{y} - s[\mathbf{C}_3\mathbf{a} + \mathbf{y}]. \tag{3.13c}$$

The boundary condition rows can then be removed from (3.13a–b) exactly as described in Section 2. Again, if $\mathbf{B}_5$ is singular, the method fails; however, in most problems it is invertible.

Recently a Chebyshev–Galerkin method has been proposed which removes spurious modes and retains the infinite-order convergence of the Chebyshev spectral methods [7]. This method requires the construction of a set of basis functions which are linear combinations of Chebyshev polynomials and which satisfy the boundary conditions. An examination of this method reveals that it involves a level of computational effort that is at least equivalent to the level required by the modified tau method proposed here. In addition, the modified tau method does not require the construction of special basis and test functions, and it treats the differential operators in a more natural fashion. Lastly, the modified tau method provides the convergence information of the tau coefficients, which is not provided by a Galerkin method. Thus the modified tau method is a useful alternative to the Chebyshev–Galerkin method proposed in [7], as demonstrated by the results of Section 4.

3.3. *Extension of the Modified Tau Method to a System of Fourth-Order Equations*

The above method can be extended to eigenvalue problems that are expressed as systems of higher-order differential equations, although the removal of the "y" vector is somewhat more difficult. To illustrate the procedure, consider the following eigenvalue problem posed as a system of two differential equations:

$$L_{11}L_1u_1+L_{12}L_2u_2+M_{11}u_1+M_{12}u_2+s[\alpha_{11}L_1u_1+\alpha_{12}L_2u_2+N_{11}u_1+N_{12}u_2]=0, \quad (3.14a)$$

$$L_{21}L_1u_1+L_{22}L_2u_2+M_{21}u_1+M_{22}u_2+s[\alpha_{21}L_1u_1+\alpha_{22}L_2u_2+N_{21}u_1+N_{22}u_2]=0 \quad (3.14b)$$

where the operators L_1, L_2, L_{ij}, M_{ij}, and N_{ij} for $i=1, 2; j=1, 2$, are linear differential operators defined and non-singular on the interval $[-1, 1]$ and where appropriate boundary conditions on u_1 and u_2 are prescribed. The quantities α_{ij}, $i=1, 2$; $j=1, 2$, are constants not all of which are zero. To use the modified tau method define two new functions v_1 and v_2 by

$$v_1=L_1u_1, \qquad v_2=L_2u_2 \quad (3.15a, b)$$

and substitute these into (3.14a, b):

$$L_{11}v_1+L_{12}v_2+M_{11}u_1+M_{12}u_2+s[\alpha_{11}v_1+\alpha_{12}v_2+N_{11}u_1+N_{12}u_2]=0, \quad (3.15c)$$

$$L_{21}v_1+L_{22}v_2+M_{21}u_1+M_{22}u_2+s[\alpha_{21}v_1+\alpha_{22}v_2+N_{21}u_1+N_{22}u_2]=0. \quad (3.15d)$$

Approximating u_i and v_i, $i=1, 2$, by truncated series of Chebyshev polynomials

$$u_i \approx \sum_{n=0}^{N+2} a_{in}T_n(x), \quad (3.16a)$$

$$v_i \approx \sum_{n=0}^{N+2} b_{in}T_n(x), \qquad i=1, 2, \quad (3.16b)$$

and substituting into (3.15a–d) will yield a system of matrix equations. Let matrix $\mathbf{B}_{ij}$ represent the action of the operator L_{ij}, $i=1$, 2; $j=1$, 2, on v_j; let matrix $\mathbf{A}_{ij}$, $i=1, 2$; $j=1, 2$, represent the action of the operator M_{ij}, $i=1$, 2; $j=1$, 2, on u_j; let matrix $\mathbf{D}_{ij}$, $i=1, 2; j=1$, 2, represent the action of operator N_{ij}, $i=1$, 2; $j=1$, 2, on u_j; let matrix $\mathbf{Q}_i$ represent the action of the operator L_i, $i=1$, 2 on u_i, $i=1$, 2; let $\hat{\mathbf{a}}_i=(a_{i0}, ..., a_{i,N+2})^{\mathrm{T}}$, let $\hat{\mathbf{b}}_i=(b_{i0},..., b_{i,N+2})^{\mathrm{T}}$, let $\mathbf{b}_i=(b_{i0}, ..., b_{i,N})^{\mathrm{T}}$, and let $\mathbf{y}_i=(b_{i,N+1}, b_{i,N+2})^{\mathrm{T}}$. Let $\tau_i=(\tau_{i1}, \tau_{i2})^{\mathrm{T}}$ and define the matrices

$$(\mathbf{B}_{kj1})_{ni}=(\mathbf{B}_{kj})_{ni},$$
$$n=0, ..., N-2; \quad i=0, ..., N; \quad k=1, 2; \quad j=1, 2, \quad (3.17a)$$

$$(\mathbf{B}_{kj2})_{ni}=(\mathbf{B}_{kj})_{n+N-1, i},$$
$$n=0, 1; \quad i=0, ..., N; \quad k=1, 2; \quad j=1, 2, \quad (3.17b)$$

$$(\mathbf{B}_{kj3})_{ni}=(\mathbf{B}_{kj})_{n+N+1, i},$$
$$n=0, 1; \quad i=0, ..., N; \quad k=1, 2; \quad j=1, 2, \quad (3.17c)$$

$$(\mathbf{B}_{kj4})_{ni} = (\mathbf{B}_{kj})_{n,\, i+N+1},$$
$$n = 0, ..., N-2; \quad i = 0, 1; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17d}$$
$$(\mathbf{B}_{kj5})_{ni} = (\mathbf{B}_{kj})_{n+N-1,\, i+N+1},$$
$$n = 0, 1; \quad i = 0, 1; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17e}$$
$$(\mathbf{B}_{kj6})_{ni} = (\mathbf{B}_{kj})_{n+N+1,\, i+N+1},$$
$$n = 0, 1; \quad i = 0, 1; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17f}$$
$$(\mathbf{A}_{kj1})_{ni} = (\mathbf{A}_{kj})_{ni},$$
$$n = 0, ..., N-2; \quad i = 0, ..., N+2; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17g}$$
$$(\mathbf{A}_{kj2})_{ni} = (\mathbf{A}_{kj})_{n+N-1,\, i},$$
$$n = 0, 1; \quad i = 0, ..., N+2; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17h}$$
$$(\mathbf{A}_{kj3})_{ni} = (\mathbf{A}_{kj})_{n+N+1,\, i},$$
$$n = 0, 1; \quad i = 0, ..., N+2; \quad k = 1, 2; \quad j = 1, 2, \tag{3.17i}$$
$$(\mathbf{Q}_{k1})_{ni} = (\mathbf{Q}_{k})_{ni},$$
$$n = 0, ..., N-2; \quad i = 0, ..., N+2; \quad k = 1, 2, \tag{3.17j}$$
$$(\mathbf{Q}_{k2})_{ni} = (\mathbf{Q}_{k})_{n+N-1,\, i},$$
$$n = 0, 1; \quad i = 0, ..., N+2; \quad k = 1, 2, \tag{3.17k}$$

with the matrices $\mathbf{D}_{kjm}$, $k = 1, 2; j = 1, 2;\ m = 1,\ 2,\ 3$, being partitions defined in terms of $\mathbf{D}_{kj}$ the same way that matrices $\mathbf{A}_{kjm}$ are defined in terms of $\mathbf{A}_{kj}$. This partitioning is shown in Fig. 2. The matrix equations are then

$$[\mathbf{B}_{111}\mathbf{Q}_1 + \mathbf{A}_{111}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{121}\mathbf{Q}_2 + \mathbf{A}_{121}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{114}\mathbf{y}_1 + \mathbf{B}_{124}\mathbf{y}_2$$
$$+ s\{[\alpha_{11}\mathbf{Q}_{11} + \mathbf{D}_{111}]\,\hat{\mathbf{a}}_1 + [\alpha_{12}\mathbf{Q}_{21} + \mathbf{D}_{121}]\,\mathbf{a}_2\} = 0 \tag{3.18a}$$
$$[\mathbf{B}_{112}\mathbf{Q}_1 + \mathbf{A}_{112}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{122}\mathbf{Q}_2 + \mathbf{A}_{122}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{115}\mathbf{y}_1 + \mathbf{B}_{125}\mathbf{y}_2$$
$$+ s\{[\alpha_{11}\mathbf{Q}_{12} + \mathbf{D}_{112}]\,\hat{\mathbf{a}}_1 + [\alpha_{12}\mathbf{Q}_{22} + \mathbf{D}_{122}]\,\hat{\mathbf{a}}_2\} = 0, \tag{3.18b}$$
$$\tau_1 = [\mathbf{B}_{113}\mathbf{Q}_1 + \mathbf{A}_{113}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{123}\mathbf{Q}_2 + \mathbf{A}_{123}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{116}\mathbf{y}_1 + \mathbf{B}_{126}\mathbf{y}_2$$
$$+ s\{\alpha_{11}\mathbf{y}_1 + \mathbf{D}_{113}\hat{\mathbf{a}}_1 + \alpha_{12}\mathbf{y}_2 + \mathbf{D}_{123}\hat{\mathbf{a}}_2\}, \tag{3.18c}$$
$$[\mathbf{B}_{211}\mathbf{Q}_1 + \mathbf{A}_{211}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{221}\mathbf{Q}_2 + \mathbf{A}_{221}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{214}\mathbf{y}_1 + \mathbf{B}_{224}\mathbf{y}_2$$
$$+ s\{[\alpha_{21}\mathbf{Q}_{11} + \mathbf{D}_{211}]\,\hat{\mathbf{a}}_1 + [\alpha_{22}\mathbf{Q}_{21} + \mathbf{D}_{221}]\,\hat{\mathbf{a}}_2\} = 0, \tag{3.18d}$$
$$[\mathbf{B}_{212}\mathbf{Q}_1 + \mathbf{A}_{212}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{222}\mathbf{Q}_2 + \mathbf{A}_{222}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{215}\mathbf{y}_1 + \mathbf{B}_{225}\mathbf{y}_2$$
$$+ s\{[\alpha_{21}\mathbf{Q}_{12} + \mathbf{D}_{212}]\,\hat{\mathbf{a}}_1 + [\alpha_{22}\mathbf{Q}_{22} + \mathbf{D}_{222}]\,\hat{\mathbf{a}}_2\} = 0, \tag{3.18e}$$
$$\tau_2 = [\mathbf{B}_{213}\mathbf{Q}_1 + \mathbf{A}_{213}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{223}\mathbf{Q}_2 + \mathbf{A}_{223}]\,\hat{\mathbf{a}}_2 + \mathbf{B}_{216}\mathbf{y}_1 + \mathbf{B}_{226}\,\mathbf{y}_2$$
$$+ s\{\alpha_{21}\mathbf{y}_1 + \mathbf{D}_{213}\hat{\mathbf{a}}_1 + \alpha_{22}\mathbf{y}_2 + \mathbf{D}_{223}\hat{\mathbf{a}}_2\}. \tag{3.18f}$$

$$\mathbf{B}_{kj} = \left[\begin{array}{c|c} \mathbf{B}_{kj1} & \mathbf{B}_{kj4} \\ \hline \mathbf{B}_{kj2} & \mathbf{B}_{kj5} \\ \hline \mathbf{B}_{kj3} & \mathbf{B}_{kj6} \end{array}\right]$$

$$\mathbf{A}_{kj} = \left[\begin{array}{c} \mathbf{A}_{kj1} \\ \hline \mathbf{A}_{kj2} \\ \hline \mathbf{A}_{kj3} \end{array}\right]$$

$$\mathbf{Q}_{k} = \left[\begin{array}{c} \mathbf{Q}_{k1} \\ \hline \mathbf{Q}_{k2} \end{array}\right]$$

FIG. 2. Partitioning of the matrices $\mathbf{B}_{kj}$, $\mathbf{A}_{kj}$, and $\mathbf{Q}_k$ in Section 3.3.

Equations (3.18b, e) are then solved simultaneously for $\mathbf{y}_1$ and $\mathbf{y}_2$:

$$\mathbf{y}_1 = -[\mathbf{E}_{11} + s\mathbf{H}_{11}]\,\hat{\mathbf{a}}_1 - [\mathbf{E}_{21} + s\mathbf{H}_{21}]\,\hat{\mathbf{a}}_2, \tag{3.19a}$$

$$\mathbf{y}_2 = -[\mathbf{E}_{12} + s\mathbf{H}_{12}]\,\hat{\mathbf{a}}_1 - [\mathbf{E}_{22} + s\mathbf{H}_{22}]\,\hat{\mathbf{a}}_2, \tag{3.19b}$$

$$\mathbf{E}_i = \mathbf{G}^{-1}\begin{bmatrix} \mathbf{B}_{1i2}\mathbf{Q}_i + \mathbf{A}_{1i2} \\ \mathbf{B}_{2i2}\mathbf{Q}_i + \mathbf{A}_{2i2} \end{bmatrix}, \qquad i = 1, 2, \tag{3.19c}$$

$$\mathbf{H}_i = \mathbf{G}^{-1}\begin{bmatrix} \alpha_{i1}\mathbf{Q}_{i2} + \mathbf{D}_{1i2} \\ \alpha_{i2}\mathbf{Q}_{i2} + \mathbf{D}_{2i1} \end{bmatrix}, \qquad i = 1, 2, \tag{3.19d}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{B}_{115} & \mathbf{B}_{125} \\ \mathbf{B}_{215} & \mathbf{B}_{225} \end{bmatrix}, \tag{3.19e}$$

$$(\mathbf{E}_{i1})_{nj} = (\mathbf{E}_i)_{nj}, \qquad n = 0, 1; \quad j = 0, ..., N+2; \quad i = 1, 2, \tag{3.19f}$$

$$(\mathbf{E}_{i2})_{nj} = (\mathbf{E}_i)_{n+2,j}, \qquad n = 0, 1; \quad j = 0, ..., N+2; \quad i = 1, 2, \tag{3.19g}$$

$$(\mathbf{H}_{i1})_{nj} = (\mathbf{H}_i)_{nj}, \qquad n = 0, 1; \quad j = 0, ..., N+2; \quad i = 1, 2, \tag{3.19h}$$

$$(\mathbf{H}_{i2})_{nj} = (\mathbf{H}_i)_{n+2,j}, \qquad n = 0, 1; \quad j = 0, ..., N+2; \quad i = 1, 2, \tag{3.19i}$$

where it is assumed that $\mathbf{G}$ is invertible. Substituting into (3.18a, d) gives the final matrix equations

$$[\mathbf{B}_{111}\mathbf{Q}_1 + \mathbf{A}_{111} - \mathbf{B}_{114}\mathbf{E}_{11} - \mathbf{B}_{124}\mathbf{E}_{12}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{121}\mathbf{Q}_2 + \mathbf{A}_{121} - \mathbf{B}_{114}\mathbf{E}_{21} - \mathbf{B}_{124}\mathbf{E}_{22}]\,\hat{\mathbf{a}}_2$$
$$+ s[\alpha_{11}\mathbf{Q}_{11} + \mathbf{D}_{111} - \mathbf{B}_{114}\mathbf{H}_{11} - \mathbf{B}_{124}\mathbf{H}_{12}]\,\hat{\mathbf{a}}_1$$
$$+ s[\alpha_{21}\mathbf{Q}_{21} + \mathbf{D}_{121} - \mathbf{B}_{114}\mathbf{H}_{21} - \mathbf{B}_{124}\mathbf{H}_{22}]\,\hat{\mathbf{a}}_2 = 0, \tag{3.20a}$$

$$[\mathbf{B}_{211}\mathbf{Q}_1 + \mathbf{A}_{211} - \mathbf{B}_{214}\mathbf{E}_{11} - \mathbf{B}_{224}\mathbf{E}_{12}]\,\hat{\mathbf{a}}_1 + [\mathbf{B}_{221}\mathbf{Q}_2 + \mathbf{A}_{221} - \mathbf{B}_{214}\mathbf{E}_{21} - \mathbf{B}_{224}\mathbf{E}_{22}]\,\hat{\mathbf{a}}_2$$
$$+ s[\alpha_{21}\mathbf{Q}_{11} + \mathbf{D}_{211} - \mathbf{B}_{214}\mathbf{H}_{11} - \mathbf{B}_{224}\mathbf{H}_{12}]\,\hat{\mathbf{a}}_1$$
$$+ s[\alpha_{22}\mathbf{Q}_{21} + \mathbf{D}_{221} - \mathbf{B}_{214}\mathbf{H}_{21} - \mathbf{B}_{224}\mathbf{H}_{22}]\,\hat{\mathbf{a}}_2 = 0 \tag{3.20b}$$

with the same τ equations (3.18c, f) as before. The boundary equations can be removed with the method described in Section 2.

For the system of N_p fourth-order equations of the form

$$\sum_{i=1}^{N_p} \{L_{ni}L_i u_i + M_{ni}u_i + s[\alpha_{ni}L_i u_i + N_{ni}u_i]\} = 0, \qquad i = 1, ..., N_p; \quad -1 < x < 1, \tag{3.21a}$$

with appropriate linear boundary conditions on the functions u_i at ± 1, define the functions v_i by

$$v_i = L_i u_i, \qquad i = 1, ..., N_p, \tag{3.21b}$$

and expand u_i and v_i as truncated series of Chebyshev polynomials as

$$u_i \approx \sum_{n=0}^{N+2} a_{in} T_n(x), \qquad v_i \approx \sum_{n=0}^{N+2} b_{in} T_n(x), \qquad i = 1, ..., N_p.$$

Then letting matrix $\mathbf{B}_{ni}$ represent the action of the operator L_{ni} on v_i, letting matrix $\mathbf{A}_{ni}$ represent the action of the operator M_{ni} on u_i, letting matrix $\mathbf{D}_{ni}$ represent the action of the operator N_{ni} on u_i and letting matrix $\mathbf{Q}_i$ represent the action of the operator L_i on u_i, and partitioning each of these matrices in the same way that the corresponding matrices were partitioned for the case $N_p = 2$ above, the general form for the modified matrix equations is

$$\sum_{i=1}^{N_p} \left\{\mathbf{B}_{ni1}\mathbf{Q}_i + \mathbf{A}_{ni1} - \sum_{j=1}^{N_p} \mathbf{B}_{nj4}\mathbf{E}_{ij}\right\} \hat{\mathbf{a}}_i$$
$$+ s\sum_{i=1}^{N_p} \left\{\alpha_{ni}\mathbf{Q}_{i2} + \mathbf{D}_{ni1} - \sum_{j=1}^{N_p} \mathbf{B}_{nj4}\mathbf{H}_{ij}\right\} \hat{\mathbf{a}}_i = 0, \qquad n = 1, ..., N_p, \tag{3.22a}$$

$$\tau_n = \sum_{i=1}^{N_p} \left\{[\mathbf{B}_{ni3}\mathbf{Q}_i + \mathbf{A}_{ni3}]\,\hat{\mathbf{a}}_i + \sum_{j=1}^{N_p} \mathbf{B}_{nj6}\mathbf{y}_j + s[\mathbf{y}_i + \mathbf{D}_{ni3}]\,\hat{\mathbf{a}}_i\right\}, \qquad n = 1, ..., N_p, \tag{3.22b}$$

$$\mathbf{y}_n = -\sum_{j=1}^{N_p} \{\mathbf{E}_{jn} - s\mathbf{H}_{jn}\}\, \hat{\mathbf{a}}_j, \qquad n = 1, ..., N_p, \tag{3.22c}$$

$$\mathbf{E}_i = \mathbf{G}^{-1} \begin{bmatrix} \mathbf{B}_{1i2}\mathbf{Q}_i + \mathbf{A}_{1i2} \\ \mathbf{B}_{2i2}\mathbf{Q}_i + \mathbf{A}_{2i2} \\ \vdots \\ \mathbf{B}_{N_p,i,2}\mathbf{Q}_i + \mathbf{A}_{N_p,i,2} \end{bmatrix}, \qquad i = 1, ..., N_p, \tag{3.22d}$$

$$\mathbf{H}_i = \mathbf{G}^{-1} \begin{bmatrix} \alpha_{i1}\mathbf{Q}_{i2} + \mathbf{D}_{1i2} \\ \alpha_{i2}\mathbf{Q}_{i2} + \mathbf{D}_{2i2} \\ \vdots \\ \alpha_{i,N_p}\mathbf{Q}_{i2} + \mathbf{D}_{N_p,i,2} \end{bmatrix}, \qquad i = 1, ..., N_p, \tag{3.22e}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{B}_{115} & \mathbf{B}_{125} & \cdots & \mathbf{B}_{1,N_p,5} \\ \mathbf{B}_{215} & \mathbf{B}_{225} & \cdots & \mathbf{B}_{2,N_p,5} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{N_p,1,5} & \mathbf{B}_{N_p,2,5} & \cdots & \mathbf{B}_{N_p,N_p,5} \end{bmatrix}, \tag{3.22f}$$

$$(\mathbf{E}_{ij})_{nk} = (\mathbf{E}_i)_{n+2(j-1),k}, \qquad i = 1, ..., N_p; \quad j = 1, ..., N_p; \quad n = 0, 1; \quad k = 0, ..., N+2, \tag{3.22g}$$

$$(\mathbf{H}_{ij})_{nk} = (\mathbf{H}_i)_{n+2(j-1),k}, \qquad i = 1, ..., N_p; \quad j = 1, ..., N_p; \quad n = 0, 1; \quad k = 0, ..., N+2, \tag{3.22h}$$

provided $\mathbf{G}$ is invertible. The boundary condition rows can be removed from (3.22a, b) using the method described in Section 2.

Notice that in the preceding development the modified method depends on the Chebyshev polynomials only through the entries in the matrices that represent the differential operators. The matrix manipulations themselves are entirely independent of the basis functions used, and hence this method can be used with any tau method.

4. Numerical Examples

In this section three fourth-order eigenvalue problems are solved using the usual Chebyshev–tau method of Section 2 and the modified Chebyshev–tau method of Section 3. The eigenvalues obtained are compared to exact or accepted eigenvalues. The convergence of the modified Chebyshev–tau method is compared to that of the usual Chebyshev–tau method, and the use of the tau coefficients as convergence indicators is demonstrated. An obvious variation of the modified Chebyshev–tau method is explored as well.

Example 1. *A Fourth-Order Eigenvalue Problem with a Third Derivative Term*

Consider the eigenvalue problem used as an example in Sections 2 and 3:

$$u'''' + Ru''' - su'' = 0, \qquad -1 < x < 1, \tag{4.1}$$

$$u(-1) = u(1) = u'(-1) = u'(1) = 0 \tag{4.2}$$

where u is the unknown function of the independent variable x, s is the eigenvalue, and R is a real parameter. The eigencondition for this problem is

$$(R^2+4s)^{1/2}\left[1 - \frac{\cosh(R^2+4s)^{1/2}}{\cosh R}\right] + \frac{2s\sinh(R^2+4s)^{1/2}}{\cosh R} = 0. \tag{4.3}$$

TABLE I

First Two Eigenvalues and Spurious Eigenvalues Generated by the Usual Chebyshev–Tau Method for Example 1, $R=0$

Truncation order, $N+4$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$	$\lvert\tau_3\rvert$	$\lvert\tau_4\rvert$
9	0.0000000[a]	0.736×10^{-8}	0.369×10^{6}	0.000	0.000
	0.0000000[a]	0.736×10^{-8}	0.369×10^{6}	0.000	0.000
	−9.8923149	0.840×10^{-11}	0.346×10^{1}	0.000	0.000
	−25.387815	0.280×10^{3}	0.189×10^{-22}	0.000	0.000
14	0.0000000[a]	0.240×10^{8}	0.000	0.000	0.000
	0.0000000[a]	0.240×10^{8}	0.000	0.000	0.000
	−9.8696046	0.534×10^{-2}	0.000	0.000	0.000
	−20.190734	0.993×10^{-14}	0.173	0.000	0.000
19	0.0000000[a]	0.176×10^{-2}	0.106×10^{10}	0.000	0.000
	0.0000000[a]	0.176×10^{-2}	0.106×10^{10}	0.000	0.000
	−9.8696043	0.516×10^{-7}	0.630×10^{-7}	0.000	0.000
	−20.190729	0.157×10^{-3}	0.143×10^{-7}	0.000	0.000
24	0.0000000[a]	0.673×10^{10}	0.224×10^{-1}	0.000	0.000
	0.0000000[a]	0.637×10^{10}	0.224×10^{-1}	0.000	0.000
	−9.8696045	0.183×10^{-8}	0.578×10^{-8}	0.000	0.000
	−20.190728	0.278×10^{-6}	0.232×10^{-6}	0.000	0.000
29	0.0000000[a]	0.164×10^{12}	0.196×10^{-1}	0.000	0.000
	0.0000000[a]	0.164×10^{12}	0.196×10^{-1}	0.000	0.000
	−9.8696043	0.995×10^{-7}	0.221×10^{-5}	0.000	0.000
	−20.190728	0.381×10^{-6}	0.729×10^{-6}	0.000	0.000
34	0.0000000[a]	0.164×10^{12}	0.196×10^{-1}	0.000	0.000
	0.0000000[a]	0.164×10^{12}	0.196×10^{-1}	0.000	0.000
	−9.8696071	0.604×10^{-6}	0.379×10^{-6}	0.000	0.000
	−20.190728	0.221×10^{-5}	0.511×10^{-5}	0.000	0.000

Exact eigenvalues: −9.8696044, −20.1907286

[a] Spurious eigenvalue.

When R is zero the problem is self-adjoint and all the eigenvalues are real and less than zero; when R is non-zero the problem is not self-adjoint and the eigenvalues are no longer real, though the real parts are negative. It may also be shown that $s = -R^2/4$ is not an eigenvalue.

Results for the solution of the problem (4.1, 2) when $R = 0$ are given in Tables I and II. In Table I the four largest eigenvalues generated by the usual Chebyshev–tau method are given for various truncation orders, along with the magnitudes of the values of the corresponding tau coefficients; the two largest exact eigenvalues, determined from eigencondition (4.3), are also given. The matrix eigenvalue problem was solved using the EISPACK driver RG on a Cray-2 supercomputer using single precision (64-bit) arithmetic. Notice that the two largest eigenvalues, which are both zero, are in fact spurious eigenvalues as clearly indicated by the increasing magnitudes of the values of the tau coefficients. Note that, although zero is a solution of eigencondition (4.3), it is not an eigenvalue for problem (4.1, 2). By comparison to the exact eigenvalues, the non-spurious eigenvalues generated by the usual Chebyshev–tau method are seen to converge to the true eigenvalues as the truncation order $N+4$ increases. The decreasing magnitudes of the values of the tau coefficients also indicate this convergence. Note that the magnitudes of the values of the tau coefficients for the spurious eigenvalues increase as the truncation order increases. Although there are four tau coefficients for each eigenvalue in the problem when the usual Chebyshev–tau method is used, the simplicity of the differential operators for this problem forces two of these to be identically zero.

TABLE II

First Two Eigenvalues Generated by the Modified Chebyshev–Tau Method for Example 1, $R = 0$

Truncation order, $N+2$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$
9	−9.8700602	0.247×10^{-13}	0.179
	−20.295078	0.356×10^{2}	0.205×10^{-24}
14	−9.8696045	0.105×10^{-3}	0.000
	−20.190730	0.425×10^{-15}	0.612×10^{-2}
19	−9.86960431	0.119×10^{-10}	0.104×10^{-9}
	−20.1907286	0.305×10^{-5}	0.127×10^{-9}
24	−9.8696044	0.595×10^{-9}	0.690×10^{-11}
	−20.190728	0.489×10^{-10}	0.146×10^{-9}
29	−9.8696047	0.411×10^{-11}	0.256×10^{-9}
	−20.190729	0.150×10^{-8}	0.258×10^{-8}
34	−9.8696151	0.357×10^{-8}	0.203×10^{-10}
	−20.190729	0.744×10^{-8}	0.906×10^{-9}

Exact eigenvalues: −9.8696044, −20.1907286

In Table II the two largest eigenvalues generated by the modified Chebyshev–tau method are given for various truncation orders, along with the magnitudes of the values of the corresponding tau coefficients. The matrix eigenvalue problem was solved using the EISPACK driver RG on a Cray-2 supercomputer using single precision (64-bit) arithmetic. No spurious eigenvalues are generated, and the largest eigenvalue converges to the exact value at least as fast as in the usual Chebyshev–tau method.

Problem (4.1, 2) was also solved for $R=4$; results are presented in Tables III and IV for the usual and modified Chebyshev–tau methods, respectively. Eigenvalues were determined using the EISPACK driver RG on a Cray-2 supercomputer using single precision (64-bit) arithmetic. Again notice the appearance of two spurious eigenvalues when the usual Chebyshev–tau method is used; these are clearly identified by the fact that magnitudes of the values of their tau coefficients increase as the truncation order increases. The complex conjugate eigenvalue pair with largest real part generated by the usual Chebyshev-tau method is seen to converge to the exact eigenvalue pair (determined from eigencondition (4.3)) by comparison with the exact values, and this convergence is clearly indicated by the decreasing magnitudes of the values of the tau coefficients for these eigenvalues (Table III). When the modified Chebyshev–tau method is used, no spurious eigenvalues are produced and the eigenvalues converge to the exact values at least as rapidly as for the usual Chebyshev–tau method (Table IV).

TABLE III

First Two Eigenvalues and Spurious Eigenvalues
Generated by the Usual Chebyshev–Tau Method for Example 1, $R=4$

Truncation order, $N+4$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$	$\lvert\tau_3\rvert$	$\lvert\tau_4\rvert$
9	$0.0000000 \pm 0.0000000i$[a]	0.151×10^{5}	0.959×10^{5}	0.000	0.000
	$-16.816664 \pm 9.0072834i$	0.840×10^{-11}	0.346×10^{1}	0.000	0.000
14	$0.0000000 \pm 0.0000000i$[a]	0.917×10^{7}	0.648×10^{8}	0.000	0.000
	$-17.912904 \pm 9.4583875i$	0.773×10^{-1}	0.928×10^{-2}	0.000	0.000
19	$0.0000000 \pm 0.0000000i$[a]	0.219×10^{8}	0.266×10^{9}	0.000	0.000
	$-17.912922 \pm 9.4584014i$	0.169×10^{-4}	0.181×10^{-5}	0.000	0.000
24	$0.0000000 \pm 0.0000000i$[a]	0.159×10^{10}	0.122×10^{9}	0.000	0.000
	$-17.912928 \pm 9.4584016i$	0.691×10^{-6}	0.116×10^{-6}	0.000	0.000
29	$0.0000000 \pm 0.0000000i$[a]	0.649×10^{9}	0.130×10^{11}	0.000	0.000
	$-17.912922 \pm 9.4584018i$	0.167×10^{-5}	0.159×10^{-6}	0.000	0.000
34	$0.0000000 \pm 0.0000000i$[a]	0.410×10^{11}	0.208×10^{10}	0.000	0.000
	$-17.912923 \pm 9.4584033i$	0.552×10^{-5}	0.264×10^{-6}	0.000	0.000

Exact eigenvalues: $-17.9129218 \pm 9.45840144i$

[a] Spurious eigenvalue.

TABLE IV

First Two Eigenvalues Generated by the Modified Chebyshev–Tau Method for Example 1, $R = 4$

Truncation order, $N+2$	Eigenvalue	$\|\tau_1\|$	$\|\tau_2\|$
9	$-15.089059 \pm 8.0168737i$	0.942×10^{1}	0.266×10^{1}
14	$-17.912916 \pm 9.4583864i$	0.117×10^{-1}	0.150×10^{-2}
19	$-17.912922 \pm 9.4584014i$	0.119×10^{-5}	0.197×10^{-6}
24	$-17.912922 \pm 9.4584015i$	0.228×10^{-8}	0.358×10^{-9}
29	$-17.912922 \pm 9.4584011i$	0.472×10^{-7}	0.669×10^{-8}
34	$-17.912926 \pm 9.4584094i$	0.168×10^{-5}	0.238×10^{-6}
Exact eigenvalues:	$-17.9129218 \pm 9.45840144i$		

An obvious variation of the modified Chebyshev–tau method presented in Section 3 is to factor only the highest order derivative in the equation—in this example, the fourth-order derivative—and retain all the lower order derivatives of the original function. In terms of this example, the system

$$v'' + Ru''' - su'' = 0, \qquad v = u'', \quad -1 < x < 1, \tag{4.4}$$

is solved along with boundary conditions (4.2). The coefficients of the expansion of v were expressed in terms of the coefficients of the expansion of u and removed, as in the modified Chebyshev–tau method presented in Section 3. The results of solving (4.4) this way are presented in Table V (the eigenvalues were determined by the EISPACK driver RG on a Cray-2 computer using single precision arithmetic). Notice that no spurious eigenvalues are generated, and the eigenvalues converge to the exact values, at about the same rate as those generated by the usual or modified

TABLE V

First Two Eigenvalues Generated by the Variation of the Modified Chebyshev–Tau Method for Example 1, $R = 4$

Truncation order, $N+2$	Eigenalue	$\|\tau_1\|$	$\|\tau_2\|$
9	$-17.945354 \pm 9.4908166i$	0.478×10^{3}	0.483×10^{1}
14	$-17.912924 \pm 9.4583902i$	0.722×10^{-2}	0.232×10^{2}
19	$-17.912922 \pm 9.4584014i$	0.685×10^{1}	0.120×10^{-6}
24	$-17.912922 \pm 9.4584015i$	0.176×10^{-8}	0.520
29	$-17.912922 \pm 9.4584019i$	0.385×10^{-2}	0.125×10^{-7}
34	$-17.912925 \pm 9.4584097i$	0.108×10^{-7}	0.784×10^{-4}
Exact eigenvalues:	$-17.9128218 \pm 9.45840144i$		

Chebyshev–tau methods. The convergence is clearly indicated by the magnitudes of the values of the tau coefficients.

Example 2. *The Orr–Sommerfeld Stability Equation for Plane Poiseuille Flow*

The Orr–Sommerfeld stability equation for plane Poiseuille flow has been solved by a variety of methods, including the usual Chebyshev–tau method [2, 6]. The equation results from assuming that a velocity disturbance of the form

$$V(x, y, t) = u(y) \exp[i\alpha(x - st)] \tag{4.5}$$

perturbs the steady pressure-induced flow $U(y) = (1 - y^2)$ between two infinite parallel plates located (in dimensionless variables) at $y = \pm 1$; the resulting linear stability equation is

$$[u'''' - 2\alpha^2 u'' + \alpha^4 u]/(-i\alpha R) + [(U - s)(u'' - \alpha^2 u) - U''u] = 0, \qquad -1 < y < 1, \tag{4.6}$$

with boundary conditions

$$u(-1) = u(1) = u'(-1) = u'(1) = 0 \tag{4.7}$$

where u is the amplitude of the velocity disturbance (defined in Eq. (4.5)), α is the wavenumber, R is the Reynolds number, the stability parameter for this problem, and U is the known steady base flow whose stability is being examined. From Eq. (4.5) it is seen that the critical eigenvalue s_c is the eigenvalue s whose imaginary part first becomes positive as R is increased from zero, because for this eigenvalue the disturbance $V(x, y, t)$ will grow exponentially with time instead of being damped out.

Problem (4.6, 7) was solved for $\alpha = 1.00$ and $R = 10{,}000$ using the usual Chebyshev–tau method, the modified Chebyshev–tau method, and the variation of the modified Chebyshev–tau method described in Example 1. The published critical eigenvalues of Orszag [2] and Zebib [6] are used for comparison with the eigen-

The usual Chebyshev-tau method solves problem (4.6, 7) directly. The eigenvalues reported in Table VI were determined using the IMSL routine EIGZC on a CDC 835 computer with single precision (60-bit) arithmetic. Two spurious eigenvalues with large imaginary parts are produced; these spurious eigenvalues are clearly recognized by the large magnitudes of the values of at least one of the tau coefficients. The convergence of the eigenvalue with largest imaginary part to the published eigenvalue is evident by comparison with the published value and is clearly indicated by the decreasing magnitudes of the values of the tau coefficients.

The modified Chebyshev–tau method solves problem (4.6, 7) as

$$Dv/(-i\alpha R) + [(U - s)v - U''u] = 0, \qquad -1 < y < 1, \tag{4.8a}$$

$$v = Du, \qquad Df \equiv f'' - \alpha^2 f, \qquad -1 < y < 1, \tag{4.8b}$$

TABLE VI

Least Stable Eigenvalues and Spurious Eigenvalues Generated by the Usual Chebyshev–Tau Method for the Orr–Sommerfeld Stability Equation (Example 2, $\alpha = 1.00$, $R = 10{,}000$)

Truncation order, $N+4$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$	$\lvert\tau_3\rvert$	$\lvert\tau_4\rvert$
14	$0.18262407 + 0.14726056i$[a]	0.220×10^{-12}	0.111×10^{4}	0.183×10^{-9}	0.925×10^{3}
	$0.17619604 + 0.10342550i$[a]	0.785×10^{3}	0.189×10^{-9}	0.653×10^{3}	0.158×10^{-9}
	$0.76629952 + 0.29302327i$	0.628×10^{-11}	0.652×10^{2}	0.613×10^{-11}	0.636×10^{2}
19	$0.12021030 + 0.68468620i$[a]	0.783×10^{-9}	0.691×10^{4}	0.523×10^{-9}	0.426×10^{4}
	$0.13966865 + 0.52230010i$[a]	0.430×10^{4}	0.472×10^{-7}	0.304×10^{4}	0.333×10^{-7}
	$0.44617093 + 0.13137698i$	0.166×10^{2}	0.692×10^{-11}	0.182×10^{2}	0.939×10^{-11}
24	$0.097799237 + 20.375413i$[a]	0.148×10^{-6}	0.189×10^{5}	0.929×10^{-7}	0.118×10^{5}
	$0.096591927 + 16.829209i$[a]	0.182×10^{5}	0.000	0.114×10^{5}	0.000
	$0.57979484 + 0.024205325i$	0.373×10^{-12}	0.664×10^{1}	0.259×10^{-12}	0.624×10^{1}
29	$0.074701104 + 48.289614i$[a]	0.000	0.553×10^{5}	0.000	0.327×10^{5}
	$0.081609752 + 41.110103i$[a]	0.398×10^{5}	0.971×10^{-7}	0.239×10^{5}	0.583×10^{-7}
	$0.23690887 + 0.0036551612i$	0.000	0.193	0.000	0.144
34	$0.065437406 + 97.556758i$[a]	0.0000	0.980×10^{5}	0.000	0.566×10^{5}
	$0.064929953 + 85.735016i$[a]	0.101×10^{6}	0.000	0.580×10^{5}	0.000
	$0.23743315 + 0.0037224798i$	0.775×10^{-1}	0.000	0.167×10^{-1}	0.000
39	$0.054299570 + 178.04720i$[a]	0.165×10^{-4}	0.214×10^{6}	0.930×10^{-5}	0.120×10^{6}
	$0.057820287 + 158.75548i$[a]	0.162×10^{6}	0.147×10^{-4}	0.920×10^{5}	0.832×10^{-5}
	$0.23752676 + 0.0037342659i$	0.354×10^{-9}	0.178×10^{-1}	0.548×10^{-9}	0.677×10^{-2}

Exact critical eigenvalue: $0.23752649 + 0.00373967i$ [2, 6]

[a] Spurious eigenvalue.

TABLE VII

Least Stable Eigenvalues Generated by the Modified Chebyshev–Tau Method for the Orr–Sommerfeld Stability Equation (Example 2, $\alpha = 1.00$, $R = 10{,}000$)

Truncation order, $N+2$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$
14	$0.52900096 + 0.22074414i$	0.598×10^{-12}	0.357×10^{2}
19	$0.73111753 + 0.096973658i$	0.317×10^{2}	0.224×10^{-9}
24	$0.24033386 + 0.0064426763i$	0.500	0.493×10^{-10}
29	$0.23757258 + 0.0037438271i$	0.155×10^{-9}	0.980×10^{-1}
34	$0.23755789 + 0.0037060033i$	0.734×10^{-1}	0.342×10^{-9}
39	$0.23752741 + 0.0037419091i$	0.812×10^{-9}	0.151×10^{-1}
Exact critical eigenvalue:	$0.23752649 + 0.00373967i$ [2, 6]		

with boundary conditions (4.7). Results are presented in Table VII. The eigenvalues were determined using the EISPACK driver CG on a Cray-2 supercomputer with single precision (64-bit) arithmetic. No spurious eigenvalues were produced, and the eigenvalue with largest imaginary part converges to the published value as is evident by comparison with the published value. The convergence is clearly indicated by the decreasing magnitudes of the values of the tau coefficients, and in this case is more rapid than that of the usual Chebyshev–tau method.

The variation of the modified Chebyshev–tau method solves problem (4.6, 7) as

$$[v'' - 2\alpha^2 v + \alpha^4 u]/(i\alpha R) + [(U - s)(v - \alpha^2 u) - U''u] = 0, \qquad v = u'', \qquad -1 < y < 1, \tag{4.9a, b}$$

with boundary conditions (4.7). Results are presented in Table VIII. The eigenvalues were determined using the IMSL routine EIGZC on a CDC 835 computer

TABLE VIII

Least Stable Eigenvalues Generated by the Variation to the Modified Chebyshev–Tau Method for the Orr–Sommerfeld Stability Equation (Example 2, $\alpha = 1.00$, $R = 10{,}000$)

Truncation order, $N+2$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$
14	$0.48120108 + 0.22955218i$	0.277×10^{-11}	0.454×10^{2}
19	$0.66622766 + 0.12235193i$	0.521×10^{2}	0.000
24	$0.68929954 + 0.026998416i$	0.123×10^{-11}	0.210×10^{2}
29	$0.23812852 + 0.0039733169i$	0.000	0.286
34	$0.23764505 + 0.0037498712i$	0.133	0.000
39	$0.23752419 + 0.0037457516i$	0.623×10^{-10}	0.250×10^{-1}
Exact critical eigenvalue:	$0.23752649 + 0.00373967i$ [2, 6]		

with single precision (60-bit) arithmetic. No spurious eigenvalues are produced. The eigenvalue with largest imaginary part converges to the published value as is evident by comparison with the published value. (Note that both the symmetric and antisymmetric eigenmodes were retained here. Since the least stable eigenmode is symmetric and the symmetric and antisymmetric modes are uncoupled, only the symmetric modes need to be considered, as was done in [2] and [6].) The convergence is clearly indicated by the decreasing magnitudes of the values of the tau coefficients, although the convergence is slower than for the usual or modified Chebyshev–tau methods.

Example 3. *A Fourth-Order Eigenvalue Problem*

Consider finally the problem

$$D(D-s)\,u=0, \qquad r_1<r<r_2, \quad r_1>0, \tag{4.10a}$$

$$u(r_1)=u(r_2)=\frac{du}{dr}(r_1)=\frac{du}{dr}(r_2)=0, \tag{4.10b}$$

$$Du\equiv\frac{d^2u}{dr^2}+\frac{2}{r}\frac{du}{dr}-\frac{l(l+1)\,u}{r^2} \tag{4.10c}$$

where l is a positive integer. This type of problem arises, for example, when the method of partial spectral expansions is used to solve linearized fluid flow problems in a spherical geometry, in which r represents the radial variable and u represents a function related to the velocity. To solve problem (4.10a–c), the independent variable r is first mapped to a new independent variable with range $[-1, 1]$, e.g.,

$$x=[2r-(r_1+r_2)]/(r_2-r_1). \tag{4.11}$$

Using mapping (4.11), problem (4.10a–c) was solved for $l=1$, $r_1=99.00$, and $r_2=100.00$ using both the usual and modified Chebyshev–tau methods. A problem similar to this has arisen in the authors' hydrodynamic stability studies in spherical shells [10]. Since the problem is self-adjoint, the eigenvalues are real, and they are also negative, as may be shown from the eigencondition. The two largest true eigenvalues generated by each method are shown in Tables IX and X for various truncation orders, along with any spurious eigenvalues generated; the eigenvalues were determined using the EISPACK driver RG on a Cray-2 supercomputer using single precision (64-bit) arithmetic. The usual Chebyshev–tau method generates two spurious eigenvalues which are clearly identified by their large magnitude and positive sign, as well as by the large magnitudes of the values of the tau coefficients (Table IX). In addition, the magnitudes of the values of the tau coefficients for the spurious eigenvalues increase with truncation order, rather than decrease. The eigenvalues do converge to the true values as may be seen by comparison with the true values (determined from the eigencondition); this convergence is clearly indicated by the decreasing magnitudes of the values of the tau coefficients.

TABLE IX

First Two Eigenvalues and Spurious Eigenvalues Generated by the Usual Chebyshev–Tau Method for Example 3 ($l = 1$, $r_1 = 99.00$, $r_2 = 100.00$)

Truncation order, $N+4$	Eigenvalue	$\|\tau_1\|$	$\|\tau_2\|$	$\|\tau_3\|$	$\|\tau_4\|$
9	0.17088508×10^5 [a]	0.224×10^5	0.628×10^8	0.351×10^5	0.969×10^2
	0.91378395×10^4 [a]	0.599×10^{10}	0.295×10^7	0.143×10^5	0.394×10^2
	-0.39478155×10^2	0.221	0.286×10^1	0.160×10^{-2}	0.441×10^{-5}
	-0.81353359×10^2	0.242×10^6	0.526×10^3	0.804	0.222×10^{-2}
14	0.11775673×10^6 [a]	0.552×10^9	0.783×10^{12}	0.281×10^9	0.753×10^6
	0.87459004×10^5 [a]	0.168×10^{10}	0.105×10^7	0.160×10^4	0.430×10^1
	-0.39477934×10^2	0.168×10^{-2}	0.417×10^{-5}	0.349×10^{-8}	0.939×10^{-11}
	-0.80762981×10^2	0.364×10^1	0.390×10^2	0.140×10^{-1}	0.375×10^{-4}
19	0.44490216×10^6 [a]	0.471×10^7	0.438×10^{11}	0.116×10^8	0.305×10^5
	0.34890472×10^6 [a]	0.109×10^{14}	0.228×10^{12}	0.948×10^7	0.250×10^5
	-0.39477935×10^2	0.134×10^{-4}	0.185×10^{-5}	0.491×10^{-7}	0.130×10^{-9}
	-0.80762982×10^2	0.130×10^{-1}	0.223×10^{-3}	0.486×10^{-7}	0.118×10^{-9}
24	0.11787879×10^7 [a]	0.373×10^{11}	0.959×10^{14}	0.201×10^{11}	0.525×10^8
	0.99407326×10^6 [a]	0.219×10^{12}	0.682×10^8	0.121×10^6	0.316×10^3
	-0.39477935×10^2	0.367×10^{-3}	0.331×10^{-6}	0.158×10^{-9}	0.451×10^{-12}
	-0.80762983×10^2	0.282×10^{-2}	0.552×10^{-1}	0.116×10^{-4}	0.304×10^{-7}

Exact eigenvalues: -39.478216, -80.762982

[a] Spurious eigenvalue.

TABLE X

First Two Eigenvalues Generated by the Modified Chebyshev–Tau Method for Example 3 ($l = 1$, $r_1 = 99.00$, $r_2 = 100.00$)

Truncation order, $N+2$	Eigenvalue	$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$
9	−39.479634	0.455×10^{-2}	0.725×10^{-1}
	−81.180351	0.296×10^{4}	0.641×10^{1}
14	−39.477812	0.426×10^{-4}	0.105×10^{-6}
	−80.762989	0.393×10^{-1}	0.483
19	−39.477812	0.378×10^{-9}	0.556×10^{-8}
	−80.762872	0.238×10^{-3}	0.542×10^{-6}
24	−39.477812	0.165×10^{-7}	0.477×10^{-11}
	−80.762981	0.475×10^{-6}	0.103×10^{-4}

Exact eigenvalues: −39.478216, −80.762982

The eigenvalues determined by the modified Chebyshev–tau method are given in Table X. No spurious eigenvalues are produced. The eigenvalues converge to the true values as may be seen by comparison with the true values; this convergence is clearly indicated by the decreasing magnitudes of the values of the tau coefficients, and the modified method converges at least as fast as the usual Chebyshev–tau method.

5. The Tau Coefficients

As demonstrated in the previous section, the tau coefficients are useful for identifying spurious eigenvalues and for indicating the degree of convergence of the tau and modified tau methods. In fact, computation of the tau coefficients is essential to the unambiguous identification of the spurious eigenvalues generated by the usual tau method, in the absence of other criteria for identifying them. For example, in Examples 1 (when $R = 0$) and 3 of the previous section, the self-adjointness of each problem ensured that the eigenvalues for each are real, and study of the eigencondition indicated that the eigenvalues for each are in fact negative; thus eigenvalues with non-zero imaginary parts or eigenvalues with positive real parts could be rejected on the basis of this information. However, in non-self-adjoint problems, the eigenvalues are in general complex, and may not all be of one sign; or the problem may be so complex that it is impossible, or at least impractical, to obtain such information about the eigenvalues. In such cases, the tau coefficients should be calculated to unambiguously identify the spurious eigenvalues [illegible]

The major disadvantage with computing the tau coefficients for an eigenvalue is that, to do so, the eigenvector corresponding to the eigenvalue must also be computed. However, the increase in computation is often justified by the convergence information provided by the tau coefficients.

It is important to note that in eigenvalue problems, the values of the tau coefficients are dependent on the eigenvector used to calculate them. Since the eigenvector is unique in direction only, the tau coefficients corresponding to a given eigenvalue and eigenvector could be made artificially large or small by multiplying the eigenvector by a constant factor: if $\mathbf{x} = (x_0, x_1, ..., x_N)^{\mathrm{T}}$ is an eigenvector corresponding to the eigenvalue s, then so is $\alpha\mathbf{x} = (\alpha x_0, \alpha x_1, ..., \alpha x_N)^{\mathrm{T}}$ for any non-zero α. The tau coefficients for the eigenvalue s are multiplied by α as well. To make the values of the tau coefficients meaningful, it is important that the eigenvector be normalized in some consistent way. An easy and useful way to do this is to divide the eigenvector by the first non-zero element or the first non-zero element whose magnitude is greater than some predetermined value. This latter method was used in the computations discussed in Section 4.

Since only the magnitudes of the values of the tau coefficients are necessary for identifying spurious eigenvalues or indicating convergence, it is often helpful to use merely the sum of the absolute values of the tau coefficients to do this. Using this quantity is valid because if any one of the magnitudes of the tau coefficients for a given eigenvalue is large, then that eigenvalue is either spurious or a poor approximation to a true eigenvalue. Using the sum of the absolute values of the tau coefficients is especially helpful when a large system of equations is being solved: if, for example, a system of N fourth-order equations is being solved, then there will be $4N$ tau coefficients for each eigenvalue; and, if any one of these is large in magnitude, then the eigenvalue is either spurious or not a good approximation to a true eigenvalue.

6. Summary and Conclusions

A modification of the usual tau method has been presented which eliminates the spurious eigenvalues produced by that method for many eigenvalue problems. The modified method essentially involves an appropriate factorization of the differential operator; it has been developed here for an eigenvalue problem posed as a single differential equation of order greater than 2 and also for a system of such equations. By solving a variety of fourth-order eigenvalue problems, including the Orr–Sommerfeld stability equation for plane Poiseuille flow, it has been demonstrated that the modified method does not generate spurious eigenvalues and converges at least as rapidly as the usual method, sometimes more rapidly. An obvious modification of the modified method is also explored; although it does not produce spurious eigenvalues, it often does not converge as rapidly as either the usual method or the modified method. The use of the often-neglected tau coefficients as identifiers of spurious eigenvalues and indicators of convergence has also been demonstrated.

APPENDIX. CHEBYSHEV POLYNOMIALS

Chebyshev polynomials, also called Chebyshev functions of the first kind, are the solutions of the Stürm–Liouville problem

$$(1-x^2)^{1/2}\frac{d}{dx}\left[(1-x^2)^{1/2}\frac{d}{dx}u\right]+n^2u=0, \qquad -1<x<1, \quad n=0, 1, ..., \tag{A1}$$

$$u(-1) \text{ and } u(1) \text{ finite.}$$

The solutions $u(x)=T_n(x)$ are given by

$$T_n(x)=\cos[n\cos^{-1}(x)]. \tag{A2}$$

The first three Chebyshev polynomials are

$$T_0(x)=1, \qquad T_1(x)=x, \qquad T_2(x)=2x^2-1,$$

and the Chebyshev polynomials satisfy the recursion relation

$$T_{n+2}(x)-2xT_{n+1}(x)+T_n(x)=0, \qquad n\geqslant 0. \tag{A3}$$

Some useful relations for derivatives are

$$\begin{aligned}(1-x^2)\,T'_n(x)&=n[T_{n-1}(x)-xT_n(x)], & n\geqslant 1,\\ &=n[xT_n(x)-T_{n+1}(x)], & n\geqslant 0,\\ &=(n/2)[T_{n-1}(x)-T_{n+1}(x)], & n\geqslant 1,\end{aligned} \tag{A4}$$

$$(1-x^2)\,T''_n(x)-xT'_n(x)+n^2T_n(x)=0, \qquad n\geqslant 0. \tag{A5}$$

Chebyshev polynomials are orthogonal in the inner product

$$\langle T_m, T_n\rangle=\int_{-1}^{1}T_m(x)\,T_n(x)(1-x^2)^{-1/2}\,dx=c_n\pi\delta_{mn}/2, \tag{A6}$$

$$c_n=\begin{cases}2, & \text{if } n=0;\\ 1, & \text{if } n=1, 2, ...\end{cases} \tag{A7}$$

Some other useful results are

$$T_n(-x)=(-1)^n\,T_n(x), \tag{(A8)}$$

$$T_n(\pm 1)=(\pm 1)^n, \tag{(A9)}$$

$$T_{2n}(0)=(-1)^n, \tag{A10}$$

$$T_{2n+1}(0)=0, \tag{A11}$$

$$T_n(x)\,T_m(x)=[T_{n+m}(x)+T_{|n-m|}(x)]/2, \tag{A12}$$

$$T_n^{(k)}(\pm 1) = (\pm 1)^{n+k} \prod_{m=0}^{k-1} (n^2 - m^2)/(2k-1)!!, \qquad k \geqslant 1, \tag{A13}$$

$$N!! \equiv \begin{cases} N(N-2)(N-4)\cdots 1, & \text{for } N \text{ odd;} \\ N(N-2)(N-4)\cdots 2, & \text{for } N \text{ even,} \end{cases}$$

where in (A11) $T_n^{(k)}$ is the kth derivative of $T_n(x)$. Most of these results follow from Eq. (A2) and trigonometric identities.

Consider now the expansion of a function $f(x)$ or its derivatives in terms of Chebyshev polynomials on the interval $[-1, 1]$. Assume that f and its derivatives can be expanded as

$$f(x) = \sum_{n=0}^{\infty} a_n^{(0)} T_n(x), \qquad \frac{d^m f}{dx^m} = \sum_{n=0}^{\infty} a_n^{(m)} T_n(x), \qquad m = 0, 1, \ldots \tag{A14}$$

Then recursion relation (A15) below gives

$$c_{n-1} a_{n-1}^{(m)} = 2n a_n^{(m-1)} + a_{n+1}^{(m)}, \qquad n \geqslant 1, \tag{A15}$$

$$c_n a_n^{(1)} = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{\infty} p a_p, \qquad n \geqslant 0, \tag{A16}$$

$$c_n a_n^{(2)} = \sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{\infty} p[p^2 - n^2]\, a_p, \qquad n \geqslant 0, \tag{A17}$$

$$c_n a_n^{(3)} = \frac{1}{4} \sum_{\substack{p=n+3 \\ p+n \text{ odd}}}^{\infty} p[p^2(p^2-2) - 2p^2n^2 + (n^2-1)^2]\, a_p, \qquad n \geqslant 0, \tag{A18}$$

$$c_n a_n^{(4)} = \frac{1}{24} \sum_{\substack{p=n+4 \\ p+n \text{ even}}}^{\infty} p[p^2(p^2-4)^2 - 3p^4n^2 + 3p^2n^4 - n^2(n^2-4)^2]\, a_p, \qquad n \geqslant 0, \tag{A19}$$

where c_n is defined by (A7).

Suppose now that a function $g(x)$ has the expansion

$$g(x) = \sum_{n=0}^{\infty} b_n T_n(x) \tag{A20}$$

on the interval $[-1, 1]$ and the product of g with the mth derivative of f has the expansion

$$g \frac{d^m f}{dx^m} = \sum_{n=0}^{\infty} e_n^{(m)} T_n(x), \qquad m \geqslant 1. \tag{A21}$$

Then the coefficients $e_n^{(m)}$ are given by

$$c_n e_n^{(m)} = \frac{1}{2} \sum_{p=-\infty}^{+\infty} c_{|p|} c_{|n-p|} a_{|n-p|}^{(m)} b_{|p|}, \qquad n \geqslant 0; \quad m \geqslant 1, \tag{A22}$$

as is shown in [2].

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